leaves *Economic* $output_t$ and flows through the parameter c before reentering the system at the front end of the feedback loop where the circle with the plus sign is located. This does not mean that something is being subtracted from the value of *Economic* $output_t$ for it to be reentered at the plus sign. Thus, the arrow leaving *Economic* $output_t$ and pointing to the parameter c is not reducing the value of *Economic* $output_t$. Rather, a measure of *Economic* $output_t$ is being taken where the feedback loop begins, and some proportion (c) of this is being reinvested in the economy. Again, restated differently, the beginning of a feedback loop does not "pull" something out of the forward path. It merely takes a measure of the value of the forward path at that point in the system so that part of that measure can be reentered elsewhere in the system.

Also note that the parameter c instantaneously summarizes a set of diminishing feedback cycles, as is the case with all static and simultaneous systems. This is the same as was described previously with respect to Figure 2.4 and parameter m. In Chapter 3, we learn how to structure the feedback process using time operators, thereby keeping track of when an output actually feeds back into the system relative to other parts of the system.

3. GRAPH ALGEBRA AND DISCRETE-TIME LINEAR OPERATORS

So far time has not played a significant role in our discussions. Structuring the relationships between the variables with respect to time within the context of a system is one of the great strengths of graph algebra. All the models that are discussed throughout the remainder of this book use graph algebra to do this. This discussion begins with explaining how graph algebra is integrated with discrete-time applications. Discrete time implies the use of difference equations, and difference equations are often appropriate for the social sciences since a great deal of social scientific data are collected in discrete intervals. Examples of this are census data, economic data, election data, and polling data (which often correspond with an electoral calendar). Differential equations are used to model continuous-time processes and are discussed later. Models can also be built using graph algebra that have both continuous and discrete parts. These are called "metered" differential equations, and they are also discussed later in this book in the context of differential equations with embedded time lags.

All the operators used in this book are linear operators (see especially Allen 1963, p. 725; see also Goldberg, 1958). This is true of the discrete-time operators as well as the continuous-time operators. What do we mean by saying that these operators are linear? The condition of linearity requires that an operation satisfy two principles (see Cortés et al., 1974, pp. 293–294). The first is the principle of homogeneity. This states that if one multiplies a constant times a variable, and then applies an operator to this product, the answer will be the same if one first applies the operator to the variable and then multiplies this result by the constant. Symbolically, this is written as

Homogeneity: *Operator*[(*Constant*)(*Variable*)] =
Constant(*Operator* [*Variable*]).

In the case of an operator of proportional transformation, we are simply multiplying by another constant. Thus, it is clear that $abY_t = baY_t$, where a and b are constants. We will want to show that the principle of homogeneity applies to the other operators presented in this book, and this is done later as those operators are introduced.

The second condition of linearity is the principle of superposition. The superposition principle is important to the study of difference and differential equations, and it is fundamental to many dynamic processes, including the superposition of states as encountered in quantum mechanics (see, e.g., Aczel, 2003, p. 85). In general, the superposition principle states that the sum or linear combination of two or more solutions to an equation is also a solution to the equation (Zill, 2005, p. 130). In the context of linear operators, applying an operator to the sum of two variables is equal to applying the operator separately to the two variables and then summing these two results. This symbolically resembles the distributive law of multiplication, in the sense that $a[X_t + Y_t] = aX_t + aY_t$. Thus, in terms of operators, we can write

Superposition: *Operator*[*Variable* 1 + *Variable* 2] =
Operator[*Variable* 1] + *Operator*[*Variable* 2].

As with the principle of homogeneity, we will want to demonstrate that the superposition principle applies to the various operators presented in this book. Again, any operator is linear if it satisfies both these principles. Moreover, since the inverse of a linear operator reverses the functioning of that linear operator such that a variable to which the linear operator is applied is returned to its original state, inverses of linear operators are also linear operators.

Delay and Advance Operators for Discrete Time

Many social phenomena occur after some delay. That is, when a stimulus is applied, a reaction transpires at a later time. To incorporate a delay in graph algebra, a delay operator is needed. This operator is written as E^{-1}, and it is read as "E inverse." Yet other social phenomena happen in anticipation of something else that either will occur in the future or is expected to occur in the future. Thus, something happens before something else takes place. This is the opposite of a delay, and an advance operator is needed in such a situation. E^1 is the advance operator. E^{-1} is the inverse of E^1. E^{-1} changes the variable Y_t to Y_{t-1}. The advance operator (E^1) changes Y_t to Y_{t+1}. For convenience, it is conventional to write E^1 as simply E without the superscript, noting that if an advance operation is required that places a system state more than one time period into the future, then the appropriate superscript will be used, the value of which will depend on the number of time periods in question. As with delays, advance operations are a commonly encountered feature of many social phenomena. For example, when people purchase stocks on the stock market in anticipation of a future rise in the value of that stock, they are acting in advance of that expected occurrence. In a different example from the sociological literature (see Mare & Winship, 1984), some young people drop out of school in anticipation of gaining employment, while others stay in school or join the military because they anticipate a difficult time obtaining a satisfactory job. Still others may drop out of school anticipating poor employment opportunities despite a greater level of education. This can produce a self-fulfilling prophecy, a problem that may disproportionately strike young and discouraged racial minorities.

Both E and E^{-1} are linear operators. Applying the principle of homogeneity, $E[aX_t] = a(E[X_t]) = aX_{t+1}$. From the principle of superposition, we have $E[X_t + Y_t] = EX_t + EY_t = X_{t+1} + Y_{t+1}$. Similarly, with respect to E^{-1}, $E^{-1}[aX_t] = a(E^{-1}[X_t]) = aX_{t-1}$ and $E^{-1}[X_t + Y_t] = E^{-1}X_t + E^{-1}Y_t = X_{t-1} + Y_{t-1}$.

Returning to the earlier example in which campaign workers are canvassing for support by knocking on doors in a neighborhood, let us consider the campaign that occurred in Baghdad, Iraq, in January 2005. During that month, Iraqi and American officials were planning to hold elections for a new Iraqi government, and parties and politicians were actively attempting to attract support from the populace. But insurgents opposed to the American presence in Iraq were warning people not to participate in the elections (see Filkins, 2005). In this situation, the attempts of the insurgents to impose their norms on the Iraqi people would occur after a delay. First, campaign workers would interact with a potential voter. The potential voter may be influenced by this and consider supporting the candidate or party. A certain proportion of these campaign contacts would be observed by informants cooperating with the insurgents. These informants would look for evidence that the person contacted by the campaign workers might be leaning toward participating in the elections. If they suspect this to be the case, the informants would report their suspicions to the insurgents, who in turn would act to intimidate the potential voter, possibly by threatening to kill the voter and/or his or her family members. Again, this intimidation would occur after a delay, since it would take time for the informants to observe the reaction of the potential voter to the campaign stimulus and then to report this reaction to the insurgents. From a system's perspective, this is a situation that can be described in the classic terms of regulation and control associated with a delayed feedback process.

Using graph algebra, this scenario can be depicted as in Figure 3.1. Here, the input of the system is C_t, which represents the canvassing contacts that the campaign workers have with the populace, whereas the system's output is the variable V_t, which represents the result of those contacts in producing voters. In Figure 3.1, the forward path is essentially the same as it was in Figure 2.1, in the sense that the input is processed by the parameter of proportional transformation, p. However, first the input is added to a feedback path to become X_1, the first state of the system. This feedback path is called a "negative feedback loop" because of the sign of the parameter m. This parameter acts proportionally to negatively transform the output of the system (X_2) before it reenters the system. Note that the delay operator (E^{-1}) is also on the feedback path. This acts to delay the action of the feedback path by one time period.

The social interpretation of the negative feedback loop presented in Figure 3.1 is straightforward. The insurgents in Iraq calibrate their intimidation activities based on the success of the campaign workers in mobilizing new voters. As the campaign workers mobilize new voters (X_2), the insurgents will increase their intimidation. The more successful the mobilization efforts of the campaign workers, the more profound the impact of the intimidation, which algebraically means that the state of the system, X_3, will be a negative number of large magnitude. This is the classic operation of most regulation and control phenomena.

In this instance, it is heuristically useful to obtain the algebraic equation for the graph algebra diagram in Figure 3.1 both by using Mason's Rule as well as by solving for the states of the system. Beginning with the states of the system,

$$X_1 = C_t + X_3,$$
$$X_2 = pX_1,$$
$$X_3 = -mE^{-1}X_2.$$

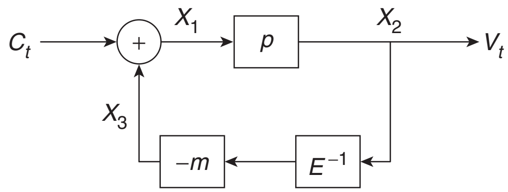

Figure 3.1 Campaign Interactions With a Negative and Delayed Feedback Loop

As before, noting that X_2 also equals V_t, we can substitute and eliminate the states of the system. After the first substitution we have $X_2 = p(C_t + X_3)$, which allows us to then write this as Equation 3.1:

$$X_2 = p(C_t - mE^{-1}X_2) \qquad [3.1]$$

Substituting for X_2, we have,

$$V_t = p[C_t - mE^{-1}V_t],$$

or, after rearranging and operationalizing the E^{-1}, we have Equation 3.2:

$$V_t = pC_t - pmV_{t-1} \qquad [3.2]$$

Since it is conventional in most of the social sciences (with the exception of economics) to write difference equations such that the lowest time script is t and the higher time scripts are t+_, we can multiply both sides of Equation 3.2 by the advance operator (E) and obtain Equation 3.3:

$$V_{t+1} = pC_{t+1} - pmV_t \qquad [3.3]$$

Equation 3.3 is a first-order linear difference equation with constant coefficients. The theory of such equations is complete (Goldberg, 1958). In economics, the convention is to write difference equations such that the highest time script is t and all lower time subscripts are t−_. If this convention was followed here, the model would have been left in the form found in Equation 3.2. Regardless, both Equations 3.2 and 3.3 are equivalent, and one can move from one to the other by multiplying through by either the advance operator, E, or the delay operator, E^{-1}, as needed. Both these operators are linear operators that obey the normal rules of algebra. Since they only work on time-dependent variables, they have no effect on constants.

We can also obtain Equation 3.3 from Figure 3.1 using Mason's Rule. This avoids having to work with the states of the system, and this is sometimes more convenient. Directly applying Mason's Rule to the graph algebra diagram in Figure 3.1, we have

$$V_t = C_t[p/(1 + pmE^{-1})].$$

Rearranging yields

$$V_t(1 + pmE^{-1}) = pC_t,$$

and then

$$V_t + pmV_tE^{-1} = pC_t.$$

Now we operationalize the delay operator, E^{-1}, to produce

$$V_t + pmV_{t-1} = pC_t.$$

We then multiply throughout by E and obtain

$$EV_t + pmEV_{t-1} = pEC_t.$$

Our final form of the model is obtained after operationalizing E and rearranging to produce $V_{t+1} = pC_{t+1} - pmV_t$, which is identical to Equation 3.3.

Including an Additive Constant With Graph Algebra

Some social scientists might want to include an additive constant with the above model that would not get involved with the feedback loop, since such an additive constant would typically be employed when estimating the model using regression. There are a number of ways that this can be accomplished using graph algebra (see, e.g., Przeworski, 1975). Two different approaches to this are discussed below, the first of which is shown in Figure 3.2. The second way is easier and is discussed later. In practice, if the second way is followed, an additive constant is sometimes omitted from the graph algebraic analysis until the model is evaluated with respect to a body of data. Both these approaches can also be used to include more than just an additive constant to the model. For example, if a researcher wishes to include a function comprising a variety of control variables, linearly combined with slopes and an intercept, this can be done as well. Again, there are other ways to include functions and additive constants to a model. The two approaches described below are simply examples of how this is commonly accomplished.

In Figure 3.2, the additive constant is inserted into the model after the feedback loop. The placement of the additive constant after the feedback loop is both substantively interesting and algebraically a bit more challenging. From

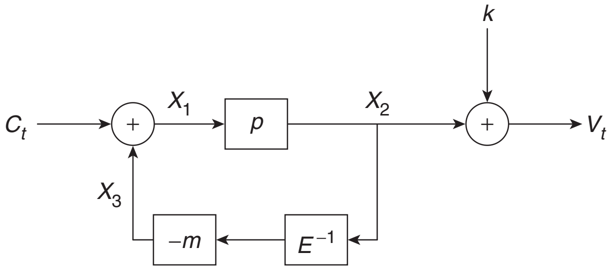

Figure 3.2 An Additive Constant Included After the Feedback Loop

a substantive point of view (continuing with the electoral campaign in Iraq example), it may be that the insurgents are responding only to personal interactions between campaign workers and the populace. That is, if the additive constant represents some additional voter mobilization process that does not originate from personal campaign interactions, then the insurgents may not react to that component of the mobilization effort. For example, the extra voter support may arise from news media broadcasts. The insurgents may not be able to monitor who is listening to the broadcasts, and therefore they may not be able to intimidate those potential voters. Thus, the feedback loop begins before the additive constant is included in the model. The algebraic complexities that result from doing this can be seen when examining the reduced-form version of the model that would ultimately be estimated using regression.

The estimated version of Equation 3.3 combined with an additive constant (omitting the error term for simplicity) is presented in Equation 3.4:

$$V_{t+1} = \beta_0 + \beta_1 C_{t+1} + \beta_2 V_t \qquad [3.4]$$

Using this form with the model shown in Figure 3.2, we have $\beta_0 = k(pm+1)$, $\beta_1 = p$, and $\beta_2 = -pm$. The formula for β_0 may at first seem strange, but it follows from the graph algebra. Return to Equation 3.1. We can no longer simply substitute V_t for X_2 since V_t now includes an additive constant that is not part of X_2. But we note that $V_t = X_2 + k$, and thus $X_2 = V_t - k$, which we can substitute into Equation 3.1. We then obtain Equation 3.5:

$$V_t - k = p[C_t - mE^{-1}(V_t - k)] \qquad [3.5]$$

Multiplying through all the brackets, we now have Equation 3.6:

$$V_t - k = pC_t - pmE^{-1}V_t + pmkE^{-1} \qquad [3.6]$$

Since m, p, and k are all constants, E^{-1} has no effect on them, and thus the delay operator can be eliminated from the last term of Equation 3.6. Rearranging, operationalizing the remaining E^{-1} with respect to V_t, and finally multiplying through by E to advance all the time scripts such that $t+1$ is the highest scripted value yields Equation 3.7:

$$V_{t+1} = pC_{t+1} - pmV_t + k(pm+1) \qquad [3.7]$$

It is clear now that the term $k(pm+1)$ is simply a constant that equals β_0 in Equation 3.4. Thus, Equation 3.7 is identical to Equation 3.3, except that Equation 3.7 includes an additive constant.

After obtaining the values of β_0, β_1, and β_2, one needs to calculate the values of the parameters p, m, and k. This is a straightforward problem of normal algebra since there are three equations and three unknowns [i.e., $\beta_0 = k(pm+1)$, $\beta_1 = p$, and $\beta_2 = -pm$]. However, the estimated parameters β_0, β_1, and β_2 are useful to us in other ways. The dynamic behavior of the model as depicted in Figure 3.2 can be determined directly through its reduced-form version, Equation 3.4. Thus, if the estimated parameter β_2 has a value between -1 and 0, then this model will display convergent oscillatory behavior over time. If the value of β_2 is less than -1, then the model will predict unstable oscillatory behavior for this system, a highly volatile outcome given the nature of the electoral dynamics modeled in this example. Other values of this estimated parameter may produce different dynamical behaviors. Readers can find complete descriptions of the dynamics of first-order linear difference equations with constant coefficients in any number of texts on finite mathematics. My personal suggestions are Goldberg (1958), Goldstein, Schneider, and Siegel (1988, chap. 11), and Baumol (1970).

Also, the equilibrium value for this model can be found by setting $V_{t+1} = V_t = V^*$, and then substituting V^* into Equation 3.4. After solving for V^*, one obtains

$$V^* = [\beta_0 + \beta_1 C_t]/(1-\beta_2),$$

which is constant only if C_t remains stationary. Otherwise, this is a "moving equilibrium value" that is driven by the value of the system's input. For this reason, a system input is often referenced as a "driver to the system." A wider discussion of difference equation models with such characteristics can be found in Huckfeldt, Kohfeld, and Likens (1982).

Now we turn to a second approach commonly used to include an additive constant in a graph algebra model. This is shown in Figure 3.3. In this figure, the additive constant is included in the model at the beginning of the feedback loop, not after the feedback loop as in Figure 3.2. From a substantive

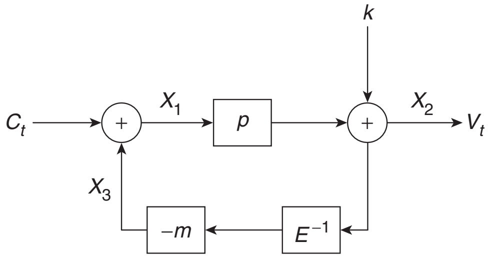

Figure 3.3 An Additive Constant Included at the Beginning of the Feedback Loop

perspective while continuing with the example of Iraqi electioneering, we may theorize that the Iraqi insurgents are keeping track of the entire vote-mobilization campaign, not just the interactions between campaign workers and potential voters. If the insurgents sense that the vote-mobilization campaign is threatening their cause through a variety of means (e.g., media broadcasts, interpersonal contacts), then they might try to suppress participation in the elections by intimidating the entire potential voting populace rather than just those interacting with campaign workers. Their strategies to broadly intimidate a populace might include more widely targeted assassinations and bombings.

Following this scenario, the repressive feedback of the Iraqi insurgents would be in response to the value of the forward path of the model plus the additive input k. This changes and simplifies the algebra when compared with the approach used in Figure 3.2. For Figure 3.3, note that $X_2 = pX_1 + k = V_t$. We begin with the statements,

$$X_1 = C_t + X_3,$$
$$X_2 = pX_1 + k,$$
$$X_3 = -mE^{-1}X_2.$$

Substituting gives us Equation 3.8, which in turn can be simplified as Equation 3.9:

$$V_t = p[C_t - mE^{-1}V_t] + k \quad [3.8]$$

$$V_{t+1} = pC_{t+1} - pmV_t + k \quad [3.9]$$

Applying Equation 3.4 as the reduced-form version of this model, we have $\beta_0 = k$, $\beta_1 = p$, and $\beta_2 = -pm$, which again gives us three equations and three unknowns.

Again, there are other ways of including an additive constant component into a graph algebra model. The first applied graph algebra model published in the *American Political Science Review* was developed by Przeworski (1975), and this model inserted the additive constant at the end of the feedback loop. That would be comparable to placing it at the summation point before X_1 in Figure 3.3. In one of my own published examples of a graph algebra model, an additive constant was employed following the summation of multiple forward paths (Brown, 1991, p. 191). In general, the placement of all graph algebra components depends solely on the social theory being addressed, and so specification variety across models will be the norm.

Difference and Summation Operators for Discrete Time

Up to this point, we have used only delay and advance operators to structure a model with respect to time. But some phenomena require models that take the difference of variables between two time points, accumulate values of variables across many time points, or both. The operators that do this are difference and summation operators. We treat these two operations together since they are the inverse of one another.

The difference operator is written as Δ. As with all time-structuring operators, Δ works only on time-scripted variables. By definition, $\Delta X_t = X_{t+1} - X_t$. To show that Δ is a linear operator, we need to apply the principles of homogeneity and superposition. Beginning with the principle of homogeneity, we note that $\Delta[aX_t] = aX_{t+1} - aX_t = a[X_{t+1} - X_t] = a(\Delta[X_t])$. Now with respect to the principle of superposition, we need to show that $\Delta[X_t + Y_t] = \Delta X_t + \Delta Y_t$. Note that $\Delta[X_t + Y_t] = [X_{t+1} + Y_{t+1}] - [X_t + Y_t] = [X_{t+1} - X_t] + [Y_{t+1} - Y_t] = \Delta X_t + \Delta Y_t$. Thus, the operator, Δ, satisfies the two conditions of linearity required of all linear operators.

It is quite common to model social processes that respond not to the value of a variable but rather to the change in that variable from one time point to the next. Let us return to our Iraqi elections example and say that the number of people deciding to participate in the elections from day to day is dependent not on the level of campaign contacts, C_t, but rather the change in campaign activity. The reason for this could be that voters will see the elections as a battle between those who want the elections to succeed and those who want them to fail. Seeing an increase in campaign activity may convince potential voters that those wanting the elections to succeed are prevailing in the campaign. This would embolden voters to participate on election day. On the other hand, if the level of campaign activity only stays the same or decreases, then voters may think that those who oppose the

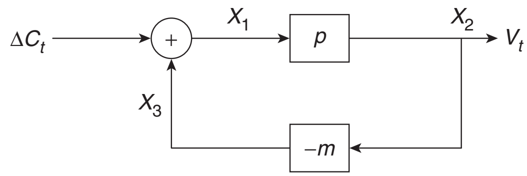

Figure 3.4 A Difference Operator in the Input to the Forward Path

elections are gaining the upper hand, dampening any sense of pro-elections momentum.

We can include the idea of change in the variable C_t in a model of the elections process as shown in Figure 3.4. In this figure, the additive constant and the delay operator have been removed to simplify the presentation. The summation operator and the delay operator will be added to the model later to show how these elements change the model's functioning.

We can use Mason's Rule to obtain the algebraic model that corresponds to the graph algebra diagram in Figure 3.4. This is shown here as Equation 3.10:

$$V_t = \Delta C_t[p/(1+pm)] \qquad [3.10]$$

Note that in Equation 3.10 the parameters p and m are both involved nonlinearly with ΔC_t. When estimated, this model would produce an overdetermined system of equations, since one slope estimate, say β_1, would equal $p/(1+pm)$. Thus, we would be trying to get two parameter values out of one estimated number, which we cannot do. This problem is not caused by the inclusion of the difference operator in the model. The problem is caused by the fact that the structure of the model does not separate the parameters p and m by time, as was done by including a delay operator in the feedback loop of Figure 3.3.

Rather than reintroduce the delay operator in the feedback loop, let us place a summation operator along the forward path of the model. For simplicity, let us also go back to the version in which the input is the number of daily or weekly campaign contacts, C_t, not change in those contacts. This is shown in Figure 3.5. The Δ^{-1} operator is a summation operator, and it is read as "delta inverse." It is located on the forward path of the model depicted in this figure and acts to accumulate or sum up the new voters who are being mobilized by the activities of the campaign workers, and a substantive reason for wanting to do this in a model is suggested below. But before diving into the substantive interpretation of the model, it is worth making a few observations regarding the functioning of the Δ^{-1} operator.

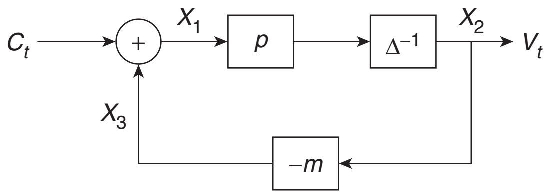

Figure 3.5 A Discrete-Time Summation Operator in the Forward Path

The two operators, Δ and Δ^{-1}, are inverse operations, in the sense that their interaction, $\Delta\Delta^{-1}$, yields the identity operation (the operator for which is I). In practice, this means that

$$\Delta\Delta^{-1}Y_t = IY_t = Y_t$$

(see Goldberg, 1958, pp. 41–44). Whereas Δ is an operator that requires that we difference a variable to yield some specified value, Δ^{-1} is an operator that requires us to find a function that will result in that same value when we difference it. Thus, if $\Delta Y_t = y_t$, where y_t is the difference and can be anything, including a constant, then $Y_t = \Delta^{-1}y_t$. The operator Δ^{-1} is the finite analog to integration with respect to the calculus. Just as integration is more challenging than differentiation in calculus, similarly performing the Δ^{-1} operation is more challenging than finding a difference using Δ. Fortunately, the property of inverse operations allows us easily to eliminate the Δ^{-1} operation in practice when using graph algebra for nearly all settings, as is explained through example below.

It is helpful to show why the operator Δ^{-1} actually accumulates over time. Let us say that Δ^{-1} transforms an input C_t into an output V_t. This can be stated as in Equation 3.11:

$$\Delta^{-1}C_t = V_t \qquad [3.11]$$

Since Δ^{-1} is the inverse of Δ, we can say that $\Delta V_t = C_t$, which is the same as $V_{t+1} - V_t = C_t$. This can be rewritten as Equation 3.12:

$$V_{t+1} = C_t + V_t \qquad [3.12]$$

Now multiply both sides of Equation 3.12 by E^{-1} to obtain Equation 3.13:

$$V_t = C_{t-1} + V_{t-1} \qquad [3.13]$$

Substitute Equation 3.13 for V_t in Equation 3.11, and you have

$$\Delta^{-1}C_t = C_{t-1} + V_{t-1}. \qquad [3.14]$$

Thus, Equation 3.14 demonstrates that the Δ^{-1} operator yields a mapping of an indefinite sum that results from having our current output (V_t, or equivalently, $\Delta^{-1}C_t$) equal to the addition of the input for the last time period (C_{t-1}) and the output for the last time period (V_{t-1}). Thus, we are taking the previous input and adding it to the previous output to get the new output. With each new iteration, we are adding one more input (from one time period only) to the previous output (which has been accumulating across all iterations), which is simply the accumulation of the input over time (see also Cortés et al., 1974, pp. 299–300).

There are many substantive reasons why we might want to model social processes using a summation operator as in Figure 3.5. Before returning to our electoral example, consider the situation of China with respect to the gender imbalance that is causing considerable stress within the population (see Yardley, 2005). There has long been a social bias favoring male children in China, which has resulted in the widespread practice of mothers opting for an abortion if the fetus is a girl. The problem is compounded by strict regulations placed on families by the Chinese government aimed at limiting population growth by reducing the number of births per couple to one. By 2005, all of this resulted in a significant imbalance between the genders, with boys outnumbering girls in some areas 134 to 100. There are long-term economic implications to this as well. The growing Chinese economy will require a stable labor force for years to come. But the reduction in Chinese fertility (defined as births per female), plus the reduced number of women available to give birth, could cause an eventual steep drop in the youthful working-age population. This gender imbalance had been slowly accumulating since the Chinese government enacted their population-control measures in the 1970s, and the government eventually reacted to the accumulated imbalance, fearing an eventual "baby bust." This is a classic situation of a feedback process that is responding not to the incremental inputs of more boys being born than girls, but to the accumulated result of these incremental inputs summed up over many years. One can use the Δ^{-1} operator to model such processes.

Returning now to an electoral example as depicted in Figure 3.5, we can say that the feedback process is responding not to the daily acts of the campaign workers interacting with potential voters, but rather the accumulated success of these contacts. This can be a general electoral process and not one relating only to the Iraqi election example used earlier. Feedback that occurs to enforce the community norms in response to a political campaign can be based on how many new voters are formed by the campaign activity over a long period of time. When the accumulated new voting support grows in the population, the community responds to repress this growth, as operationalized by the parameter $-m$, which appears in the feedback loop.

The system defined in Figure 3.5 can be expressed algebraically using Mason's Rule, as is shown in Equation 3.15:

$$V_t = C_t[(p\Delta^{-1}/(1+pm\Delta^{-1})] \qquad [3.15]$$

Rearranging Equation 3.15 yields $V_t(1+pm\Delta^{-1}) = C_t p\Delta^{-1}$, which then simplifies to $V_t + pm\Delta^{-1}V_t = C_t p\Delta^{-1}$. Rearranging this gives us Equation 3.16:

$$V_t = C_t p\Delta^{-1} - pm\Delta^{-1}V_t \qquad [3.16]$$

At this point, we want to eliminate the Δ^{-1} in Equation 3.16. We do so by multiplying both sides of the equation by Δ, which gives us $\Delta V_t = C_t p\Delta\Delta^{-1} - pm\Delta\Delta^{-1}V_t$. Since Δ and Δ^{-1} are inverse operations, they cancel each other, leaving us with the algebraically tractable statement found in Equation 3.17:

$$\Delta V_t = C_t p - pmV_t \qquad [3.17]$$

Operationalizing the Δ operator as $\Delta V_t = V_{t+1} - V_t$ and then rearranging Equation 3.17 gives us our final form of the model, stated here as Equation 3.18:

$$V_{t+1} = pC_t + V_t[1-pm], \qquad [3.18]$$

a first-order linear difference equation with constant coefficients.

It is useful to compare Equation 3.18 with Equation 3.3 (from Figure 3.1), which is repeated for convenience below:

$$V_{t+1} = pC_{t+1} - pmV_t \qquad [3.3]$$

Note that both the summation operator and the delay operator placed somewhere within a system containing feedback produce a first-order time-structured system. The placement of the variables in Equations 3.18 and 3.3 is the same, in the sense that both C_t and V_t appear on the right-hand sides of both equations (although note the difference in time scripts for the variable C_t in both equations). But the arrangement of the parameters is different for both equations, and this reflects differences in the social theory underlying the separate models.

It is now especially useful to see what happens if one places both a summation operator and a delay operator in the system at the same time. This could easily be justified from a substantive point of view. If the feedback reacts to the accumulation along the forward path, and if there is a delay in the activity of the feedback, we could specify our model as in Figure 3.6. In our campaign example, this might mean that the norms of the neighborhood would act to repress the success of the campaign activity after a delay. In

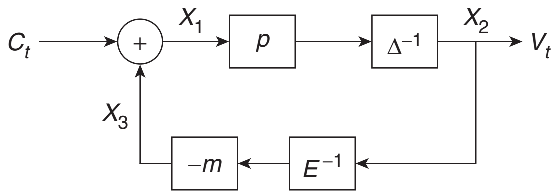

Figure 3.6 A Discrete-Time Summation Operator in the Forward Path Combined With a Delay Operator on the Feedback Path

practical terms, neighborhood organizations would respond with their own campaign activities that would be aimed at counteracting the original vote-mobilization campaign, but these response activities would occur only after the accumulated successes of the original campaign are noted by neighborhood leaders. There would be a delay in the response since it would take the neighborhood leaders time to organize their response efforts.

Figure 3.6 can be phrased algebraically as Equation 3.19 using Mason's Rule:

$$V_t = C_t[p\Delta^{-1}/(1 + pmE^{-1}\Delta^{-1})] \qquad [3.19]$$

Working through the algebra of Equation 3.19 is a process similar to that used for Equation 3.15, with the exception of the inclusion of the E^{-1} operator. Rearranging Equation 3.19 yields Equation 3.20:

$$V_t + pmE^{-1}\Delta^{-1}V_t = C_t p\Delta^{-1} \qquad [3.20]$$

Rearranging, multiplying through by Δ, and then operationalizing the E^{-1} results in Equation 3.21:

$$\Delta V_t = pC_t - pmV_{t-1} \qquad [3.21]$$

Now we operationalize the Δ on the left-hand side, rearrange, and then finally multiply through by E to advance all the time scripts to obtain our final version of the model, shown as Equation 3.22:

$$V_{t+2} = pC_{t+1} + V_{t+1} - pmV_t \qquad [3.22]$$

Equation 3.22 is a second-order linear difference equation with constant coefficients. Thus, by placing both a summation operator and a delay operator within the same system loop, we increased the order of the model by one.

In Equation 3.22, the term pC_{t+1} acts to drive the second-order system in V_t. If the campaign activity stops such that C_t goes to zero, then the output will begin to decay gradually, as is determined by the second-order system

$V_{t+2} = V_{t+1} - pmV_t$. This is an aspect of system response (see, in particular, Cortés et al., 1974, Part 3).

A Note Regarding Additive Constants

In the previous section, I introduced the idea of including an additive constant in a model using graph algebra. Since it is possible to write a model such that an isolated constant may interact with a time operator, it is worth summarizing the observations made earlier regarding how some common time operators affect isolated constants. For example, when a delay or advance operator (E or E^{-1}) combines with an isolated constant (i.e., without the multiplicative presence of a time-dependent variable), then the isolated constant is not affected by the time operator. In such a situation, the time operator can be ignored. For example, $E^{-1}r = r$, where r is a constant. On the other hand, the difference operator, Δ, yields a zero when it interacts with an isolated constant, since constants do not change with respect to time. Thus, $\Delta r = 0$.

An Estimated Example: Labor Union Membership

It is natural to ask how one would estimate complex models that are developed using graph algebra. As can be seen with the simple examples in this chapter, the use of graph algebra can very quickly lead to models with nonlinearities in the parameters, in the sense that parameters often get multiplied by (or are otherwise combined with) other parameters. More complex graph algebra models than those shown in this chapter are also easily derived, and it should be clear that the standard approach to the linear regression model where one separable parameter exists for each independent variable in a list often will not work.

There are, indeed, a number of ways to estimate such complex models. The approaches vary in difficulty from extremely simple to quite challenging. The approach that is best for any given situation depends on the complexity of the model and, to some extent, the researcher's commitment to finding estimates for the model's parameters. If the model is highly nonlinear and complex, but it has the potential to make a big impact on a given audience, then a researcher will want to invest more effort in estimating the parameters. Sometimes this will require significant programming, and some readers may want to examine some of my own efforts in this regard (Brown, 1991, 1995a). However, very often the matter of estimating graph algebra models can be handled with little more effort than with ordinary linear regression. I work through one such example here using two separate approaches to ordinary least squares.

Let us say that we are interested in the campaign to recruit new members to labor unions that occurred in the United States from 1930 through 1970, which covers the most significant period of labor union growth in the 20th century. The data for this period are shown in Table 3.1 and are obtained from the *Historical Statistics of the United States* (*Millennial Edition*). In the example worked out here, the output is labor union membership, and the input is the number of workers involved in work stoppages as a percentage of the total labor force. I explain the rationale behind using these variables in the model below. I do not use the variable for the number of work stoppages in the model, although I include it in Table 3.1 so that readers can see how the number of stoppages corresponds with the number of workers involved in the stoppages.

If one is interested in arguing that growth in labor union membership is a first-order process, where new growth comes from an expansion of the existing pool of unionized workers, then one might begin with a statistical model of an autoregressive process. This can be accomplished as with Equation 3.23. The error term is omitted for simplicity.

$$Labor_{t+1} = \beta_0 + \beta_1 Labor_t \qquad [3.23]$$

The graph algebra representation of this model is shown as Figure 3.7. Note that there are no variable inputs to the left of the system in Figure 3.7. That is, nothing outside of $Labor_t$ is driving it. The output is simply responding to itself over time via a feedback process.

We can see how well this model fits the data using a simple bivariate regression. The scatterplot of labor union membership on the lag of labor union membership is shown in Figure 3.8. For this plot, R^2 is 0.9, $\beta_0 = 1.55$, and $\beta_1 = 0.94$. Using these values and a reasonable initial condition, we can "shoot" a difference equation through the data, as is shown in Figure 3.9.

But now let us model this as a system with an input. Let us say that the growth of the labor union movement was a direct response to the activity of union members who were activists. One measure of the number of union members who are the most active in union functions is the working population that is involved in work stoppages, here measured as a percentage of the total work force. These people will be located in picket lines, for example, and many of them will be active in trying to encourage other workers to support their cause. We can think of these people as similar to the campaign activists in our previous voter-mobilization model. Using a standard statistical approach, we can incorporate this input into the graph algebra as in Figure 3.10. Our model now becomes

$$Labor_{t+1} = \beta_0 + \beta_1\ Labor_t + \beta_2\ Activists_{t+1}. \qquad [3.24]$$

TABLE 3.1
Labor Union Membership in the United States From 1930 Through 1970

Year	*Labor Union Members*[a]	*Work Stoppages*[b]	*Activitists*[c]
1930	6.8	183	0.8
1931	6.5	342	1.6
1932	6.0	324	1.8
1933	5.2	1,170	6.3
1934	5.9	1,470	7.2
1935	6.7	1,120	5.2
1936	7.4	789	3.1
1937	12.9	1,860	7.2
1938	14.6	688	2.8
1939	15.8	1,170	3.5
1940	15.5	577	1.7
1941	17.7	2,360	6.1
1942	17.2	840	2.0
1943	20.5	1,980	4.6
1944	21.4	2,120	4.8
1945	21.9	3,470	8.2
1946	23.6	4,600	10.5
1947	23.9	2,170	4.7
1948	23.1	1,960	4.2
1949	22.7	3,030	6.7
1950	22.3	2,410	5.1
1951	24.5	2,220	4.5
1952	24.2	3,540	7.3
1953	25.5	2,400	4.7
1954	25.4	1,530	3.1
1955	24.7	2,650	5.2
1956	25.2	1,900	3.6
1957	24.9	1,390	2.6
1958	24.2	2,060	3.9
1959	24.1	1,880	3.3
1960	23.6	1,320	2.4
1961	22.3	1,450	2.6
1962	22.6	1,230	2.2
1963	22.2	941	1.1
1964	22.2	1,640	2.7
1965	22.4	1,550	2.5
1966	22.7	1,960	3.0
1967	22.7	2,870	4.3
1968	23.0	2,649	3.8
1969	22.6	2,481	3.5
1970	22.6	3,305	4.7

a. Labor union membership as a proportion of total labor force.
b. Number of work stoppages.
c. Number of workers involved in work stoppages as a percentage of total labor force.

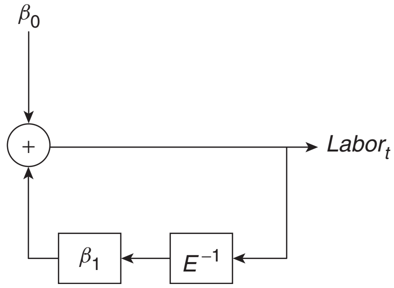

Figure 3.7 Graph Algebra of a Simple Autoregressive Process

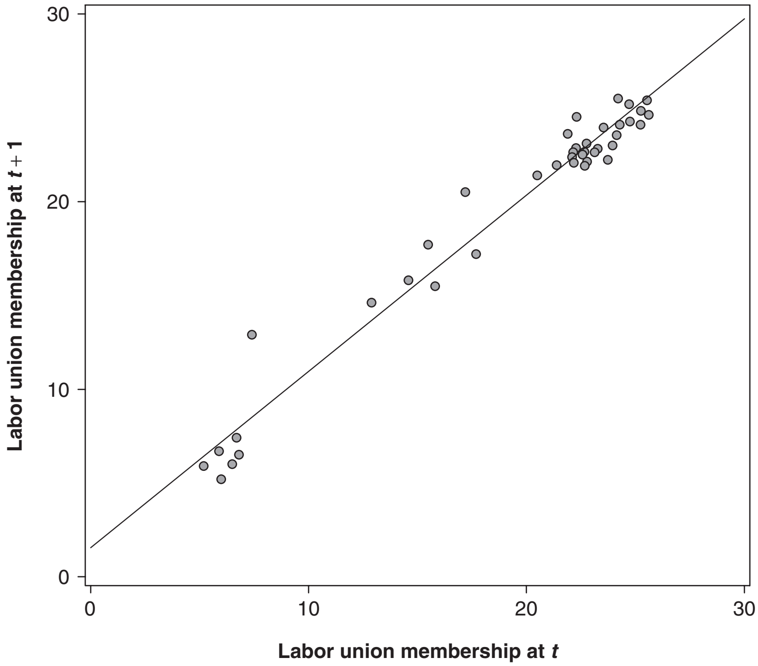

Figure 3.8 First Differences of Union Membership in the U.S. Labor Force

Let us now model this process as we did in the voter-mobilization example developed earlier, but with a different twist in the feedback path. Borrowing from the earlier discussion, we can say that the success of the labor union

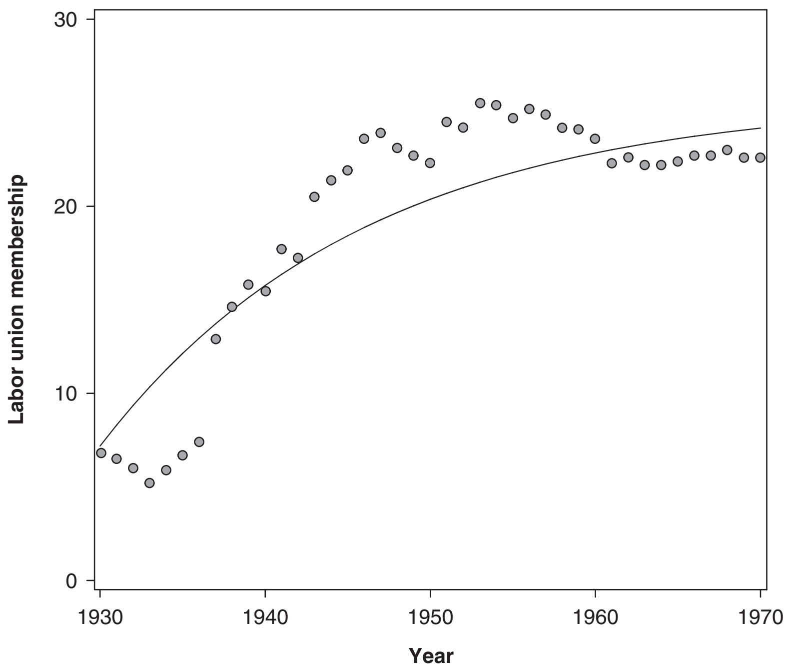

Figure 3.9 Union Membership in the U.S. Labor Force

organizing efforts will depend on the number of activists that are out there on the street trying to gain the attention of other workers. Thus, the activists themselves will be a true input that will be transformed proportionally (with respect to their own numbers) into new labor union members. But as labor union organizing efforts continue to make inroads, some proportion of those newly activated will join with the other activists to encourage workers to mobilize as well. Some of this will happen simply by the fact that the workers not yet mobilized will see the ranks of the mobilized workers growing, and they will want to join the mobilized ranks because of this. Thus, we will have a positive feedback loop, which is different from what we had in, say, Figure 3.3. This can now be depicted using graph algebra as in Figure 3.11.

The graph algebra for this model yields Equation 3.25:

$$Labor_{t+1} = \beta_0 + pmLabor_t + pActivists_{t+1} \qquad [3.25]$$

The reduced-form version of Equation 3.25 is still Equation 3.24, but now we need to determine the values of parameters p and m. By comparing

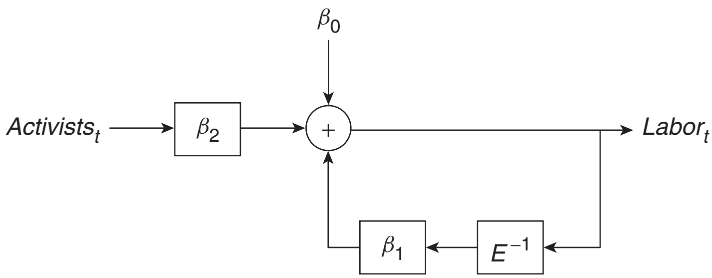

Figure 3.10 Graph Algebra of a Simple Reduced-Form Autoregressive Process

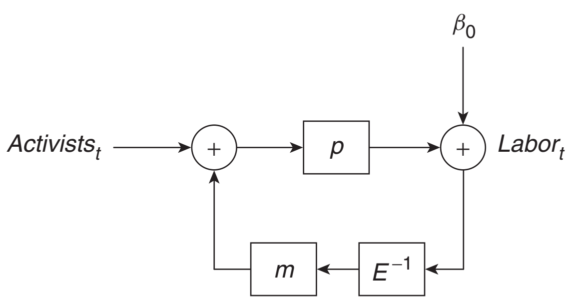

Figure 3.11 Graph Algebra of a Simple Autoregressive Process

Equations 3.24 and 3.25, we note that $\beta_1 = pm$ and $\beta_2 = p$. If we run Equation 3.24 in a multiple regression, we find that $\beta_1 = 0.944$ and $\beta_2 = 0.227$. Thus, we can now solve for the parameter on our feedback loop by dividing 0.944 by 0.227, and $m = 4.159$. The fit for this model is 0.97.

Our remaining problem is that while we do have the point estimates for our parameters, we do not yet have standard errors for those point estimates. It is important to include statistical tests for the estimated parameter values in nearly all situations. Thus, we have to use an estimation program that is different from a normal linear regression program to get the parameter estimates as well as their standard errors (and the associated P values). In many situations such as this, one can use PROC MODEL in SAS to perform this analysis easily using ordinary least squares. The exact statements that would give all the relevant statistical outputs using PROC MODEL would be as follows:

```
PROC MODEL DATA = LABOR;
ENDOGENOUS LABORUN;
EXOGENOUS LLABORUN STOPPC YEAR;
PARMS P M B;
LABORUN = (P*STOPPC) + (P*M*LLABORUN) + B;
FIT LABORUN/OLS OUT = LABOROUT OUTPREDICT;
```

In the above code, LABOR is the data set that contains the variables shown in Table 3.1, LABORUN is the unionized work force as a proportion of the total work force, LLABORUN is the lag of LABORUN, and STOPPC is the number of workers involved in work stoppages as a percentage of the total labor force. This model produces a predicted path as shown in Figure 3.12.

This example is quite simple, and researchers would normally want to specify a model of union membership growth more fully. But the basics of how to estimate many such models (or at least one approach to this) should now be clear. Another aspect that is worth mentioning is that researchers will want to check for (at least) first-order autocorrelation within the error term of the model. If this is discovered, there are different philosophies of what to do about it. One approach is to deal with it statistically (see, e.g., Ostrom, 1990). Such approaches often involve transforming the variables such that the autocorrelation is eliminated, and there are some situations in which this may be an appropriate plan of attack. But some view this approach as potentially comparable to eliminating the evidence at the scene of a crime (see Brown, 1991, pp. 203–205). If autocorrelation is present in the error term, then something systematic is escaping the model. Another way of looking at this is that autocorrelation is a sign of specification error. In situations in which one has tried all reasonable specification possibilities, then correcting the autocorrelation problem with a statistical approach may be the only option. But since graph algebra offers so much flexibility in terms of coming up with new and innovative algebraic formulations for a model, some theorists who work with graph algebra prefer to go back to the drawing board, so to speak, when faced with autocorrelation in the error term. In such situations, statistical approaches to dealing with autocorrelation are avoided, and new model specifications are derived that more effectively capture all of the systematic components that exist in the data.

With respect to working with reduced-form models, not all graph algebra models can be boiled down to a reduced-form version suitable for relatively easy estimation. But when this does happen, and if one estimates a reduced-form version of a model and then uses the parameter values for that reduced-form version (e.g., the βs in Equation 3.24) to derive the model's true parameters that are embedded within the reduced-form parameters (as with

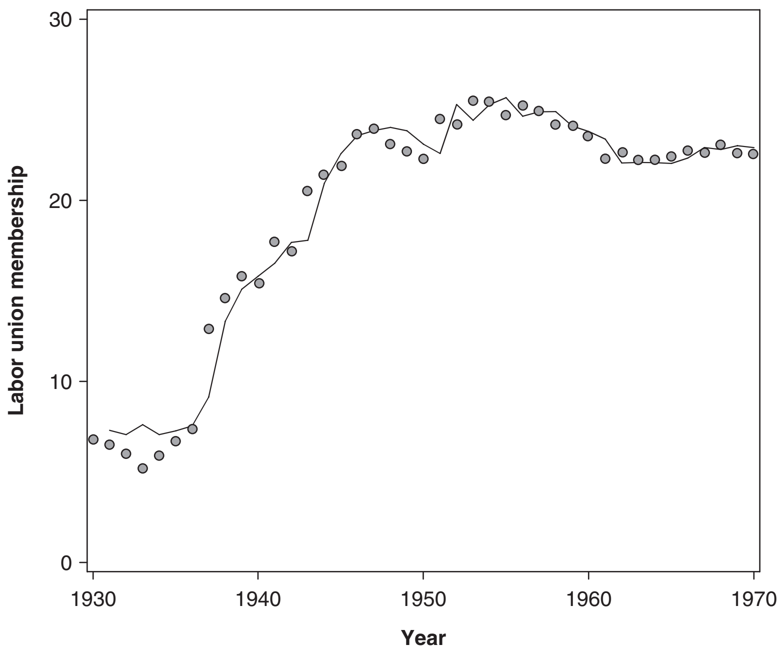

Figure 3.12 Union Membership Driven by Activists

Equation 3.25), it may be that more than one graph algebra specification can be reduced to the given reduced form. In fact, we have already seen this happen more than once in this chapter. For example, the graph algebra of Figure 3.10 produces the same reduced-form model as the one shown in Figure 3.11. Compare also Equations 3.7 and 3.9. Which one is correct when the reduced form is the same?

This is impossible to determine statistically. The correct model in this instance needs to come from social and political theory, not numbers. One works with graph algebra to translate the best theory possible into a mathematical form. After this is done, the model is sometimes reducible to a form that can be estimated using a commonly available statistical package. But as with all such models, it is the original model that is isomorphic to the theory that is being tested, and it is with the defense of that original model that a researcher would normally place his or her efforts. In a situation in which the graph algebra model stands on its own and is not reducible, the problem of competing interpretations vanishes. We will return to this issue when we examine multiple equation systems, such as Richardson's arms race model.